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On the classical similarity solutions of the continuity equation for electrons in microwave-afterglow plasmas

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Abstract. In a recent work by the author, a model for the continuity equation for electrons in microwave-afterglow plasmas has been suggested. This equation reflects the dependence of the ambipolar diffusion and of the dissociative recombination coefficients on the number density of electrons. Here, the Lie group analysis is used to generate classical similarity solutions of this equation. It is found that the general family of the group of characteristic trajectories not only leads to a classical similarity solution, already found by other ways, but also reveals a special class of similarity solutions. The existence and uniqueness for the solution of the ordinary differential equations obtained are proved. A qualitative behaviour of this solution is analysed. Also, some special forms of these equations are exactly solved. It is found that the solution of the continuity equation for the number density of electrons in the experiments carried out by Penetrante *et al* and Hdang *et al* is decreasing or increasing.

1. Introduction

The nonlinear electron continuity equation with isotropic and radially symmetric diffusion coefficient is written as

$$\partial_t N_e = x^{1-m}\partial_x(x^{m-1}D(x,t)\partial_x N_e) - \alpha N_e^2 \tag{1.1}$$

where N_e, t and x denote the number density of electrons, time and space, respectively. In microwave-afterglow plasma, D and α are the ambipolar diffusion and recombination coefficients, respectively. The values $m = 1$, 2 and 3 correspond to plasma regimes with plane, cylindrical and spherical geometry, respectively. In (1.1) the ambipolar diffusion coefficient is given by

$$D = D_0\left(1 + \frac{T_e^*}{T_e}\right) \tag{1.2}$$

where D_0 is a constant and $T_e^*(T_e)$ is the electron temperature (for a Maxwellian distribution) [1].

It has been shown in [2, 3] that α obeys a power law $\alpha = \alpha_0 T_e^{-k}$, where α_0 and $0 < k < 1$ are constants, while the same behaviour was found in [1] in different experiments but for constants $-1 < k < 1$. In view of the results reported in [1–3], a model for the variation of D with N was proposed in [4]: $D = D_0^* N_e^\gamma$ if $-1 < \gamma < 1$ and D_0^* is equal to the diffusion coefficient if $\gamma = 0$. This was based on the observation that T_e^* and T_e vary slowly with N_e (cf figures 3 and 7 of [1]). Also, a model for the variation of α with N_e was proposed as $\alpha = \alpha_0^* N_e^\beta$ where $-1 < \beta < 1$ and α_0^* is equal to the recombination coefficient when $\beta = 0$ [4]. In this situation the nonlinear electron continuity equation in microwave-afterglow plasmas can be written as

$$\partial_t N_e = D_0^* x^{1-m} \partial_x (x^{m-1} N_e^\gamma \partial_x N_e) - \alpha_0^* N_e^{\beta+2} \tag{1.3}$$

where $|\beta| < 1$ and $|\gamma| < 1$ correspond to the results found in the experiments [1–3]. Hereafter, we shall study equation (1.3) for all values of $-\infty < \beta, \gamma < \infty$. It is known in the theory of partial differential equations (PDE) that if $\beta = \gamma = 0$, equation (1.3) is classified as 'almost' linear [5] and it is nonlinear when $\beta \neq 0$ or $\gamma \neq 0$. For physical fitness, we recall some terminology for plasma regimes described by equation (1.3). This is done according to the different values of β and γ. Also, we bear in mind that the diffusion coefficient depends on γ and the recombination coefficient depends on β.

We recall that if $|\gamma| < 1$ ($|\gamma| > 1$) and $|\beta| < 1$ ($|\beta| > 1$) then equation (1.3) describes a plasma regime of weak (strong) diffusivity and weak (strong) nonlinearity, respectively. Thus, we have four possible plasma regimes to be analysed. The reason for choosing the limiting values of $|\beta| = 1$ is that (as will be shown) the value $\beta = -1$ separates two classes of classical similarity solutions. Similar argument holds for the values of $|\gamma| = 1$. Here, we shall be concerned with the boundary value problem ($\beta \neq -1$):

$$N_e(0,t) = N_0 C_0 t^s \qquad x^{m-1}\partial_x \downarrow N_e(x,t) \downarrow 0 \quad \text{as } x \downarrow 0, \quad t > 0 \tag{1.4}$$

where C_0, N_0 and s are constants and will be specified later. When $\beta = -1$, the first condition in (1.4) is replaced by

$$N_e(0,t) = N_0 e^{-s\tau}. \tag{1.5}$$

The reason for considering the conditions (1.4-5) will be explained later. We need to point out that the first condition in (1.4) and (1.5) may hold in a plasma regime with plane geometry ($m = 1$), but it may or may not hold in plasma regimes with cylindrical ($m = 2$) or spherical geometry ($m = 3$). In these regimes, it may hold that $N(0,t) \uparrow \infty$. Accordingly, the second condition in (1.4) does not hold. This boundary value problem needs a separate study.

We will demonstrate how Lie group analysis can be introduced to generate similarity solutions of (1.3). We proceed to this by performing the transformations

$$N = \frac{N_e}{N_0} \qquad \tau = \alpha_0^* N_0^{\beta+1} t \qquad X = \left(\frac{\alpha_0^* N_0^{\beta+1-\gamma}}{D_0^*}\right)^{1/2} x.$$

Now (1.3) becomes

$$\partial_\tau N = X^{1-} \partial_X (X^{m-1} N^\gamma \partial_X N) - N^{\beta+2}. \tag{1.6}$$

Under these transformations, the boundary conditions (1.4) become ($\beta \neq -1$)

$$N(0,\tau) = \tau^s, X^{m-1}\partial_X N(X,\tau) \downarrow 0 \qquad X \downarrow 0, \tau > 0 \tag{1.7}$$

where C is taken as $C = (\alpha_0^* N_0^{\beta+1})^s$. Condition (1.5) will be considered later.

2. Lie group analysis

By using the invariance groups (or Lie symmetries [6, 7] of PDEs) or by using the geometric theory of PDEs based upon the equivalent sets of differential forms [8], we obtain similarity solution of (1.6) below. To construct the group of symmetries for (1.6), we rewrite it as a coupled system of first-order PDEs in the Cauchy–Kowaleski form:

$$p = \partial_X N \tag{2.1}$$

$$q = \partial_\tau N \tag{2.2}$$

$$q = X^{1-m}\partial_X(X^{m-1}N^\gamma p) - N^{\beta+2}. \tag{2.3}$$

The infinitesimal Lie-point transformations are of the form:

$$\begin{aligned} X' &= X + \varepsilon V^X + O(\varepsilon^2) \\ \tau' &= \tau + \varepsilon V^\tau + O(\varepsilon^2) \\ N' &= N + \varepsilon V^N + O(\varepsilon^2) \\ p' &= p + \varepsilon V^p + O(\varepsilon^2) \\ q' &= q + \varepsilon V^q + O(\varepsilon^2) \end{aligned} \tag{2.4}$$

where ε is a small parameter, and the superscript h in the function V^h distinguiches different functions $V^h(x, \tau, N, p, q)$. We search for functions V^h such that the system of equations (2.1)–(2.3) is invariant under the transformations (2.4) up to first order in ε. Here, we use the standard technique presented in [6]. After some manipulation, we find that the invariance of equations (2.1)–(2.3) under the transformations (2.4) gives rise to the solutions

$$V^x = \mu a X \qquad V^\tau = \mu\tau + \nu \qquad V^N = -c\mu N \qquad V^p = -d\mu p \qquad V^q = -\mu e q \tag{2.5}$$

where

$$a = \frac{\beta-\gamma+1}{2(1+\gamma)} \qquad c = \frac{1}{1+\beta} \qquad d = \frac{\beta-\gamma+3}{2(1+\gamma)} \qquad e = \frac{\beta+2}{2(1+\gamma)} \qquad \beta \neq -1 \tag{2.6}$$

and μ and ν are arbitrary constants when $m = 2$ or 3. The general isovector field of the Lie analysis is given by

$$\tilde{V} = V^X\partial_X + V^\tau\partial_\tau + V^N\partial_N + V^q\partial_q. \tag{2.7}$$

By using (2.5) and (2.6), the isovector $\tilde{V}$ can be decomposed into ($\tilde{V}_1$, $\tilde{V}_2$) as

$$\tilde{V} = \mu\tilde{V}_1 + \nu\tilde{V}_2 \tag{2.8}$$

where

$$\tilde{V}_1 = aX\partial_X + \tau\partial_\tau - cN\partial_N - \mathrm{d}p\partial_p - eq\partial_q \tag{2.9}$$

$$\tilde{V}_2 = \partial_\tau \tag{2.10}$$

and the commutation relation $[\tilde{V}_1, \tilde{V}_2] = \tilde{V}_2$ holds.

If $m = 1$, we have $V^X = \mu a X + \lambda$, and the corresponding isovector $\tilde{V}$ can be decomposed into

$$\tilde{V} = \mu\tilde{V}_1 + \nu\tilde{V}_2 + \lambda\tilde{V}_3 \qquad \tilde{V}_3 = \partial_X \tag{2.11}$$

where $\tilde{V}_1$ and $\tilde{V}_2$ are given by (2.9), (2.10). In this case, the following commutation relation hold $[\tilde{V}_3, \tilde{V}_2] = 0$. The relation (2.11) reflects the fact that equation (1.6) is invariant under translation in the space variable when $m = 1$. Now, the group of invariant functions under infinitesimal Lie point transformations satisfies the relation

$$F(X', \tau', N', p', q') = F(X, \tau, N, p, q) \tag{2.12}$$

for some mapping F. By using (2.4) and expanding the left hand side of (2.12) up to $O(\varepsilon)$, we find the invariance condition as

$$(V^X\partial_X + V^\tau\partial_\tau + V^N\partial_N + V^p\partial_p + V^q\partial_q)F(X, \tau, N, p, q) = 0. \tag{2.13}$$

The characteristic curves of (2.13) satisfy

$$\frac{\mathrm{d}X}{V^X} = \frac{\mathrm{d}\tau}{V^\tau} = \frac{\mathrm{d}N}{V^N} = \frac{\mathrm{d}p}{V^p} = \frac{\mathrm{d}q}{V^q}. \tag{2.14}$$

In (2.14) V^X is taken either equal to $\mu a X$ or to $\mu a X + \lambda$. We confine ourselves to considering the case of $V^X = \mu a X$ for all values of $m = 1, 2, 3$.

The integration of (2.14) yields the functional invariants

$$F = X\tau^{-a} \qquad N\tau^c \qquad p\tau^d \qquad \tau^e \tag{2.15}$$

where $\tau = t + \nu/\mu$.

The similarity solutions are found through one of the following implicit function problems.

(i) Similarity solutions of the form

$$N\tau^c = \psi(X\tau^{-a},\ p\tau^d,\ q\tau^e). \tag{2.16}$$

In terms of Ovsiannikov's definitions [9], such solutions have rank 3.

(ii) Of the form

$$N\tau^c = \psi(X\tau^{-a},\ p\tau^d) \qquad q\tau^e = \tilde{\psi}(X\tau^{-a}, p\tau^d) \tag{2.17}$$

or

$$N\tau^c = \psi(X\tau^{-a},\ q\tau^e) \qquad p\tau^d = \tilde{\psi}(X\tau^{-a}, q\tau^e). \tag{2.18}$$

These solutions are of rank 2.

(iii) Of the form

$$N\tau^c = \psi(X\tau^{-a}) \qquad p\tau^d = \tilde{\psi}(X\tau^{-a}) \qquad q\tau^e = \tilde{\tilde{\psi}}(X\tau^{-a}) \tag{2.19}$$

and they are rank 1. The last solutions are called classical similarity solutions while the forms (i) and (ii) are called partially invariant solutions after the terminology of Ovsiannikov [9].

We notice that classical similarity solutions can also be obtained by using a dimensional analysis of equation (1.6) but the solutions of the forms (i) and (ii) cannot. Also, they may be obtained by the extended separation of variables. Non-classical similarity solutions of (1.6) will be studied in future work in view of the results in [10–11]. Further, similarity transformations of (1.6) in the absence of the term $N^{\beta+2}$ are special cases of those found here.

3. Classical similarity solutions

We focus our attention on the solution $N\tau^c = \psi(X)\tau^{-a}$. By writting $z = X\tau^{-c}$, we have

$$N = \tau^{-c}\psi(z). \tag{3.1}$$

Solution (3.1) explains the reason for choosing the first condition in (1.4) so that when substituting (3.1) into (1.7) one finds that $\psi(0) = 1$ and $N(0) = \tau^{-s}$. Hence, in (3.1) we take $c = s$. When substituting into (1.6), we find that ψ satisfies the equation

$$-c\psi - az\psi' = z^{1-m}(z^{m-1}\psi^{\gamma}\psi')' - \psi^{\beta+2} \qquad \beta \neq -1. \tag{3.2}$$

The boundary conditions (1.4) become

$$\psi(0) = 1 \qquad \text{and} \qquad z^{m-1}\psi'(z) \downarrow 0 \quad \text{as } z \downarrow 0. \tag{3.3}$$

This equation has been previously derived by the author by a straightforward method [4]. It appears to the author to defy obtaining an explicit solution to (3.2), except for some special values of β, γ and m. However, a qulitative behaviour of the solution of (3.2), (3.3) can be described. We proceed to this by using the following theorem.

Theorem 1. There exist $\gamma_0(\beta, m)$, $\gamma_1(\beta, m)$ and $\gamma_2(\beta, m)$ such that for $\gamma \geqslant \gamma_0(\beta, m)$, $\beta + 1 > 0$; $\gamma \geqslant \gamma_1$, $-1 < \beta + 1 < 0$ and $\gamma < \gamma_2$, $\beta + 1 < -1$, a unique positive solution of (3.2–3) on $[0, \infty[$ exists. Details of $\gamma_0(\beta, m)$, $\gamma_1(\beta, m)$ and $\gamma_2(\beta, m)$ are given by

$$\gamma_0(\beta, m) = \begin{cases} \frac{(m-2)(\beta+1)-2}{m} & \text{for } m = 1 \\ \max\left(-\beta - 2, \frac{(m-2)(\beta+1)-2}{m}\right) & \text{for } m = 2, 3 \end{cases} \tag{3.4}$$

$$\gamma_1(\beta, m) = \left\{ \frac{(\beta+1)(m+2(\beta+1))-2}{m} \right. \qquad \text{for } m = 1, 2, 3 \tag{3.5}$$

$$\gamma_2(\beta, m) = \begin{cases} \frac{(\beta+1)(m+2(\beta+1))-2}{m} & \text{for } m = 1 \\ \min\left(-\beta - 2, \frac{(\beta+1)(m+2(\beta+1))-2}{m}\right) & \text{for } m = 2, 3. \end{cases} \tag{3.6}$$

Proof. To prove the existence of a positive solution, assume that $\underline{\psi} = \mathrm{e}^{-\mu_0 z^2}$ and $\bar{\psi} = \mathrm{e}^{\mu_1 z^2}$; $\mu_i = \mu_i(\gamma, \beta, m) > 0$, $i = 0, 1$. It is clear that $\underline{\psi}$ and $\bar{\psi}$ satisfy the conditions (3.3). where $\underline{\psi}$ is a lower solution and $\bar{\psi}$ is an upper solution to (3.2)–(3.3). A direct analysis implies that a positive solution exists and $\underline{\psi} < \psi < \bar{\psi}$. We show that the solution exists under the conditions stated above. To this end, we set $w = z^m$, multiply (3.2) by $\psi^{\beta+1}$ and integrate on $[0, \infty[$, to obtain

$$\frac{ma}{\beta+2} w\psi^{\beta+2} \Big|_0^\infty + \left(\frac{ma}{\beta+2} - c\right) \int_0^\infty \psi^{\beta+2}\,\mathrm{d}w = m^2\psi^{\beta+\gamma+1}w^{2-2/m}\psi' \Big|_0^\infty$$
$$- m^2(\beta+1) \int_0^\infty \psi^{\beta+\gamma}\psi' w^{2-2/m}\,\mathrm{d}w - \int_0^\infty \psi^{2\beta+3}\,\mathrm{d}w. \tag{3.7}$$

We distinguish between the following cases.

(i) If ψ is decreasing ($\psi' < 0$) on $]0, \infty[$, we assume that $\psi \propto w^{-r}$ as $w \uparrow \infty$, $r > 0$, then equation (3.7) holds if $\beta + 2 > 0$ and $r > r_0$,

$$r_0 = \begin{cases} \dfrac{1}{\beta+2} & m = 1 \\ \dfrac{1}{\beta+2} & \beta+\gamma+2 > 0,\ m = 2 \\ \max\left(\dfrac{1}{\beta+2}, \dfrac{1}{3(\beta+\gamma+2)}\right) & \beta+\gamma+2 > 0,\ m = 3. \end{cases} \tag{3.8}$$

In these conditions and by using the second condition in (3.3), the first term in the right and left hand sides of (3.7) vanishes. If $\beta + 1 > 0$, equation (3.7) gives rise to

$$(\frac{ma}{\beta+2} - c) \int_0^\infty \psi^{\beta+2}\, \mathrm{d}w < 0$$

and then we have

$$\frac{ma}{\beta+2} - c < 0.$$

By substituting for a and c from (2.6), we find that

$$\gamma > \frac{m-2}{m}(\beta+1) - \frac{2}{m}.$$

By using the conditions (3.8) we obtain (3.4). Now, if $-2 < \beta < -1$ in (3.7), we find

$$(\frac{ma}{\beta+2} - c) \int_0^\infty \psi^{\beta+2}\, \mathrm{d}w > -\int_0^\infty \psi^{2\beta+3}\, \mathrm{d}w > -\int_0^\infty \psi^{\beta+2}\, \mathrm{d}w \tag{3.9}$$

which gives

$$\gamma > \gamma_1, \quad \gamma_1 = \frac{(\beta+1)(m+2(\beta+1)) - 2}{m}. \tag{3.10}$$

(ii) If ψ is increasing ($\psi' > 0$) on $]0, \infty[$, we assume that $\psi \propto w^{r}$ as $w \uparrow \infty$, $r > 0$, then (3.7) holds if $\beta + 2 < 0$ and $r > r_1$,

$$r_1 = \begin{cases} \dfrac{1}{|\beta+2|} & m = 1 \\ \dfrac{1}{|\beta+2|} & \beta+\gamma+2 < 0,\ m = 2 \\ \max\left(\dfrac{1}{|\beta+2|}, \dfrac{1}{3|(\beta+\gamma+2)|}\right) & \beta+\gamma+2 < 0,\ m = 3. \end{cases} \tag{3.11}$$

By using these conditions in (3.7), we obtain inequalities (3.9) and find that $\gamma > \gamma_1$ for $-1 < \beta + 1 < 0$, or $\gamma < \gamma_2$ for $\beta + 1 < -1$. By using (3.10) and the conditions in (3.11), we obtain equations (3.5), (3.6).

(iii) If ψ' changes sign on $]0, \infty[$, then there exists a point $0 < w_0 < \infty$ such that $\psi'(w_0) = 0$. If $\psi' < 0$ on $]w_0, \infty[$, we obtain $\gamma \geqslant \gamma_0$; $\beta > -1$. But if $\psi' > 0$ on $]w_0, \infty[$, we obtain the conditions $\gamma \geqslant \gamma_1$; $-2 < \beta < -1$ and $\gamma < \gamma_2$; $\beta < -2$. Next let $\gamma \downarrow \gamma_0$. Since $\psi(w, \gamma_n)$ is decreasing, we have

$$\|\psi(w,\gamma_n)\| = \sup_{0<w<\infty} \psi(w,\gamma_n) \text{ is uniformly bounded.}$$

Returning to (3.2), we find that $\|\psi''(w, \gamma_n)\|$ is also uniformly bounded. Consequently, the sequence $\{\psi(w, \gamma_n)\}$ is equicontinuous and there exists a uniformly convergent subsequence which we denote by $\{\psi^*(w, \gamma_n)\}$. We can take $\psi^*(w, \gamma_n) = \psi(w, \gamma_n)$.

Thus there exists $\psi^*(w, \gamma_0)$ such that $\psi(w, \gamma_n) \xrightarrow{n\uparrow\infty} \psi^*(w, \gamma_0)$ uniformly with respect to the prescribed norm. It remains to show now that $\psi^*(w, \gamma_0)$ satisfies (3.2)–(3.3). Since $\psi/(w, \gamma)\varepsilon C^2[0, \infty]$ satisfies (3.2)–(3.3) for all n, then by taking $n \uparrow \infty$ as the limit confirms the above statement. Similar proof holds for $\gamma \uparrow \gamma_1$.

Next, we prove the uniqueness. To this end we differentiate (3.2)–(3.3) with respect to γ and obtain

$$\begin{aligned} -c\psi_\gamma + \frac{m - w\psi'}{2(\beta+1)} - maw\psi'_\gamma = m^2\{w^{2-2/m}[\gamma\psi^{\gamma-1}\psi_\gamma\psi' + \psi^\gamma(\psi'_\gamma + \psi'\ln\psi)]\}' \\ - (\beta+2)\psi^{\beta+1}\psi_\gamma \end{aligned} \tag{3.12}$$

where the prime in (3.12) denotes the differentiation with respect to $w = z^m$, and ψ_γ satisfies the boundary conditions

$$\psi_\gamma(0) = 0 \qquad w^{2-2/m}\frac{\mathrm{d}}{\mathrm{d}w}\psi_\gamma \downarrow 0 \text{ as } w \downarrow 0. \tag{3.13}$$

When we substitute from (3.3) and (3.12) into (3.11) we find that $\psi''_\gamma w^{2-2/m} \downarrow 0$ as $w \downarrow 0$.

By successive differentiation of (3.12) with respect to w, we can prove that $w^{2-2/m}\psi_\gamma^{(n)} \uparrow 0$ as $w \uparrow 0$ for all $n \in N$ and $m = 1, 2$. This implies the uniqness of the positive solution to (3.2)–(3.3). The proof of uniqueness when $m = 3$ needs a little alteration. The proof of the theorem is complete.

The domain of existence of solutions of (3.2)–(3.3) is shown in figure 1 for $m = 1, 2, 3$. The qualitative behaviour of the solutions of (3.2)–(3.3) is presented in theorem 2.

Theorem 2. If $\psi \in C^2$ ($[0, \infty[$) is a solution of (3.2)–(3.3), then

(i) If $\beta + 1 - \frac{2}{m} < \gamma < \beta + 1$, $\beta + 1 > 0$, then ψ is increasing on $]0, \infty[$.

(ii) If $\beta + 1 < \gamma < \frac{(m+2)}{m}\beta + 1$, $\beta + 1 > 0$ and $\frac{(m+2)}{m}\beta + 1 < \gamma < \beta + 1 - \frac{2}{m}$, $\beta + 1 < 0$, then ψ is decreasing on $]0, \infty[$.

(iii) If γ belongs to the domain of existence of the solution of (3.2)–(3.3) and does not belong to that mentioned in (i) or in (ii), then ψ is increasing, decreasing or ψ' changes sign on $]0, \infty[$.

Proof. (i) We assume the converse, namely that there exists a point $0 < w_0 < \infty$ with $\psi'(w_0) < 0$. We integrate (3.2) on $[0, w]$ and use (3.3) to find

$$(ma - c)\int_0^w \psi\,\mathrm{d}w - maw_0\psi(w_0) = mw_0^{2-2/m}\psi^\gamma(w_0)\psi'(w_0) - \int_0^w \psi^{\beta+2}\,\mathrm{d}w < 0. \tag{3.14}$$

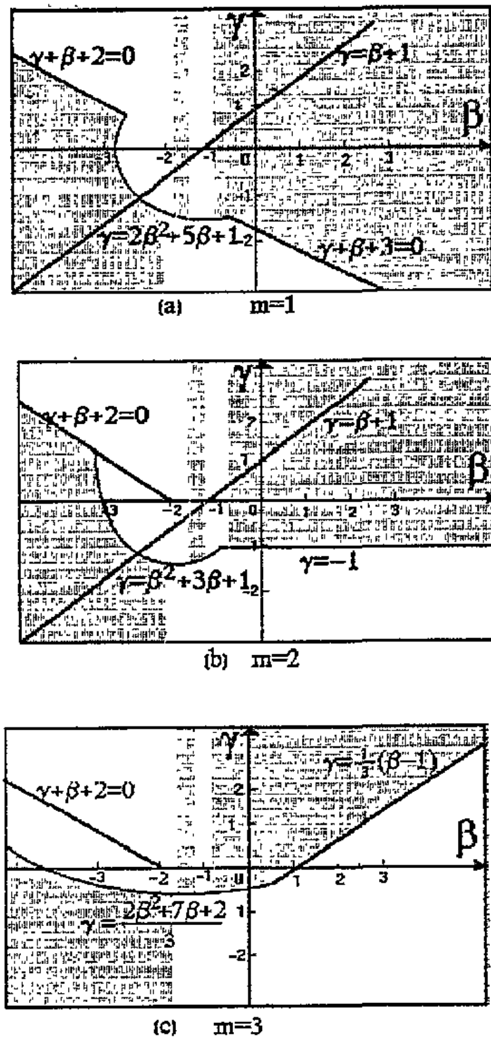

Figure 1. The lined domain in the $\beta\gamma$-plane is the domain of existence of solution of (3.2)–(3.3). (*a*) If $m = 1$, this domain is defined by $\gamma \geqslant -\beta - 3$, $\beta > -1$, $\gamma \geqslant 2\beta^2 + 5\beta + 1$, $-2 < \beta < -1$, $\gamma < 2\beta^2 + 5\beta + 1$, $-\frac{3}{2} - \frac{\sqrt{3}}{2} < \beta < -2$ and $\gamma < -\beta - 2$, $\beta < -\frac{3}{2} - \frac{\sqrt{3}}{2}$. (*b*) If $m = 2$, this domain is defined by $\gamma \geqslant -1$, $\beta > -1$, $\gamma \geqslant \beta^2 + 3\beta + 1$, $-2 < \beta < -1$, $\gamma < \beta^2 + 3\beta + 1$, $-3 < \beta < -2$ and $\gamma < -\beta - 2$, $\beta < -3$. (c) If $m = 3$, this domain is defined by $\gamma \geqslant \frac{1}{3}(\beta - 1)$, $\beta > -1$, $\gamma > (2\beta^2 + 7\beta + 2)/3$, $-2 < \beta < -1$ and $\gamma < \frac{1}{3}(2\beta^2 + 7\beta + 2)$, $\beta < -2$.

In (3.14), we used the fact that $\psi'(w_0) < 0$; therefore, (3.14) gives rise to the inequality

$$(ma - c)\int_0^w \psi \, \mathrm{d}w < maw_0\psi(w_0). \tag{3.15}$$

(a) If $a \leqslant 0$, then (3.15) holds only if $ma - c < 0$. By using (2.6) and analysing these conditions according to $\beta + 1 > 0$ or $\beta + 1 < 0$, we find that (3.15) holds if $\gamma \geqslant \beta + 1$, $\beta + 1 > 0$ and if $\gamma < \beta + 1 - \frac{2}{m}$, $\beta + 1 < 0$.

(b) If $ma - c \leqslant 0$, then (3.15) holds only if $a > 0$.

Again, we use (2.6) and analysing the above conditions we find that (3.15) holds if $\gamma < \beta + 1 - \frac{2}{m}$, $\beta + 1 > 0$ or if $\gamma > \beta + 1$, $\beta + 1 < 0$. By collecting the above results we find that the converse statement holds if $\gamma > \beta + 1$ or $\gamma \leqslant \beta + 1 - \frac{2}{m}$ when $\beta + 1 > 0$ and if $\gamma < \beta + 1 - \frac{2}{m}$ or $\gamma > \beta + 1$ when $\beta + 1 < 0$. This contradicts the assumption in (i). The statement in (i) is thus proved. The proof of (ii) is similar to that done in proving (i) while (iii) holds from the proof of (i) and (ii).

The results of theorem 2 are shown in figure 2 in the β–γ plane. The results of theorems 1 and 2 are discussed in view of the different plasma regimes proposed previously. Our aim is to determine the behaviour of the solution of (3.2), (3.3) for a given plasma regime. In a plasma regime of weak diffusivity and weak nonlinearity which corresponds to $|\gamma| < 1$ and $|\beta| < 1$, the solution of (3.2)–(3.3) exists for $m = 1, 2, 3$. The number density of electrons is either decreasing or increasing. In the last case, solutions of (3.2)–(3.3) are physically accepted if the total number of electrons remains finite. This case was also considered in [4]. However, attention was paid to obtaining decreasing solutions of (3.2). Solutions were found by using methods of approximation. In the other different plasma regimes, solutions of (3.2)–(3.3) may or may not exist. If they exist, they are also either decreasing or increasing.

3.1. Solution of special forms of (3.2)–(3.3)

In (3.2), we set $a = 0$ ($\gamma = \beta + 1$) and $m = 1$. In this case (3.2) becomes

$$-c\psi = (\psi^{\beta+1}\psi') - \psi^{\beta+2}. \tag{3.16}$$

After setting $K = \psi^{\beta+2}$ in (3.15), it integrates to

$$K'^2 = (\beta+2)\left[K^2 - \frac{2(\beta+2)}{(\beta+1)(\beta+3)}K^{(\beta+3)/(\beta+2)} + D_0\right] \qquad \beta \neq -1, -2, -3 \tag{3.17}$$

where D_0 is a constant. By using (3.3), we find that

$$D_0 = 1 + \frac{2(\beta+2)}{(\beta+1)(\beta+3)}. \tag{3.18}$$

After theorem 1, the solution of (3.17) exists for $\gamma = \beta+1$ (cf figure 1). Also, one can prove this by showing that the right-hand side of (3.17) is non-negative for all $\beta \neq -1, -2, -3$. Further, the values of K which make the RHS of (3.16) non-negative can be estimated according to the different values of β. Equivalently, the behaviour of the solution is determined. One finds that it is increasing when $\beta > 0$ and is increasing, decreasing or ψ' changes sign when $\beta < 0$. When $\beta = 0$, the solution of (3.15) is $\psi(z) = 1$. In this case, one finds $\psi(0) = \ldots = \psi^{(n)}_{(0)} = \ldots = 0$.

Now, we use a particular value $\beta = -3/2$ into (3.17) and find that the solution is

$$F\left[2\tan^{-1}\frac{\sqrt[4]{2}}{\sqrt{2}}\sqrt{\psi^{1/2}-1}\sqrt{\frac{1}{2} - \frac{5}{16\sqrt{2}}}\right] = \sqrt[4]{2}\,z \tag{3.19}$$

where $F(\gamma, \mu)$ is an elliptic integral of the first kind. When $\beta = -1/2$, the solution of (3.17) is

$$\int_{\psi^{1/2}}^{1} \frac{y^2\,\mathrm{d}y}{y^6 - \frac{12}{5}y^5 + \frac{7}{5}} = \frac{z}{\sqrt{6}}. \tag{3.20}$$

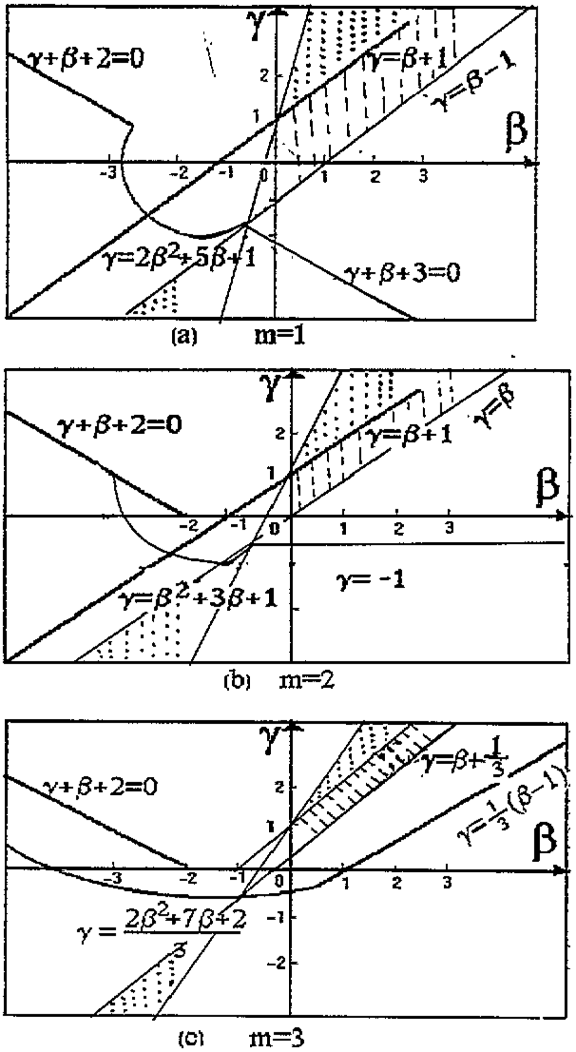

Figure 2. The dashed domain is the domain where the solution of (3.2)–(3.3) is increasing. The dotted domain is the domain where the solution of (3.2)–(3.3) is decreasing. Otherwise, in the white domain, contained in the domain of existence, the solution is increasing, decreasing or ψ' changes sign.

Note that solution (3.19) is increasing but (3.20) is decreasing.

Next we consider the particular values $\beta = -2,\ -3$. When $\beta = -3$, the integral of (3.16) and (3.3) gives

$$K'^2 = 1 - K^2 + \ln K. \tag{3.21}$$

Here, $K'^2 \geqslant 0$ implies $K \leqslant 1$ ($\psi \geqslant 1$), and in this case the solution of (3.15) is also increasing.

If $\beta = -2$, the integral of (3.15) together with (3.2) give

$$\frac{K'^2}{2} = e^K + K - 1. \tag{3.22}$$

Equation (3.22) implies $K \geqslant 0$ ($\psi \geqslant 1$) and the solution of (3.16) is also increasing.

Now, returning to (3.2), it can be written in the form

$$-ma(w^s\psi)' = m^2 w^{s-1}(w^{2-2/m}\psi^\gamma\psi') - w^{s-1}\psi^{\beta+2} \qquad \beta \neq -1 \tag{3.23}$$

where $s = \frac{c}{ma}$. If we take $\beta = -2$ and $s = 1$ ($\gamma = \frac{-2}{m} - 1$) then (3.23) integrates to

$$w\psi = -m\psi^{-1-2/m}\psi w^{2-2/m} + w \tag{3.24}$$

where the constant of integration is equal to zero by using (3.3). Equation (3.24) integrates to

$$\frac{-z^2}{2m} = \int_1^\psi \frac{y^{-\frac{2}{m}-1}}{y+1}\,\mathrm{d}y. \tag{3.25}$$

Also, equation (3.23) integrates when $s = \frac{2}{m}$ and $\beta = -2$ (or $\gamma = -3$ and $\beta = -2$). When using (3.3), we find

$$\psi' - \frac{z^2}{2}\psi^3 + (2-m)\psi - \frac{z^2}{2}\psi^2 = 0 \tag{3.26}$$

which is an Abel equation of the second kind. We notice that when setting $m = 2$ in (3.26), we obtain (3.24).

After a sequence of transformations, $\psi = h^{-1}$, $h = gz^{2-m}$ and $K = g - \frac{z^m}{2m}$, equation (3.26) takes the form

$$\frac{\mathrm{d}Z}{\mathrm{d}K} + 2Kz^{3-2m} = -\frac{1}{m}z^{3-m}. \tag{3.27}$$

Equation (3.27) is Bernoulli's equation for $m = 1, 2$.

4. Similarity solution of (1.6) for $\beta = -1$

We consider the case of $\beta = -1$ in (1.6), and find other invariants with $\mu = 0$ (of (2.5)–(2.6)) namely:

$$F = X\mathrm{e}^{\frac{-\gamma\lambda}{2}\tau} \qquad Ne^{-\lambda\tau} \qquad pe^{-\lambda(2-\gamma)\tau} \qquad qe^{-\lambda\tau} \tag{4.1}$$

where $\lambda \neq 0$ is an arbitrary constant and $\gamma \neq 0$. Here we consider the classical similarity solution:

$$Ne^{-\lambda\tau} = \psi(Z) \qquad Z = X\mathrm{e}^{\frac{-\gamma\lambda}{2}}\tau. \tag{4.2}$$

When substituting from (4.2) into (1.5), we find that $\psi(0) = 1$ and $\lambda = 5/\alpha_0$. When substituting from (4.2) into (1.6), we obtain

$$(\lambda+1)\psi - \frac{m\gamma\lambda}{2}w\psi' = m^2(w^{2-2/m}\psi^\gamma\psi')' \tag{4.3}$$

where $w = Z^m$. We solve (4.3) under the conditions

$$\psi(0) = 1 \qquad w^{2-2/m}\psi' \downarrow 0 \qquad \text{as } w \downarrow 0. \tag{4.4}$$

Theorem 3. For all $\lambda \neq 0$ or $\gamma \neq 0$ if $m = 1, 2, 3$ and $\lambda + 1 + \frac{\lambda\gamma m}{2} \leqslant 0$, $\gamma + 1 \geqslant 0$ or $\lambda + 1 + \frac{\lambda\gamma m}{2} > 0$, $\gamma + 1 < 0$ if $m = 2, 3$, a unique positive solution of (4.3)–(4.4) exists.

The proof of this theorem is similar to that of theorem 1. The domain of existence of the solution of (4.3)–(4.4) is shown in figure 3.

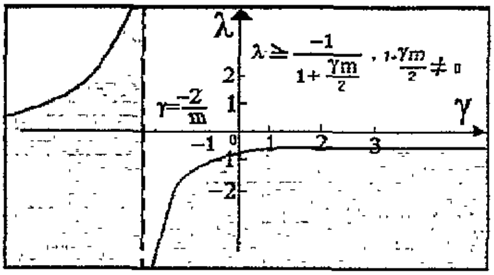

Figure 3. The lined domain in the $\gamma\lambda$-plane is the domain of existence of solution of (4.3)–(4.4).

4.1. Solution of some special forms of (4.3)–(4.4)

We rewrite equation (4.3) in the form

$$\frac{-\gamma\lambda m}{2}(w^s\psi)' = m^2 w^{s-1}(w^{2-2/m}\psi^\gamma\psi')' \tag{4.5}$$

where $s = -\frac{2(\lambda+1)}{\gamma\lambda m}$. Equation (4.5) integrates if $s = 1$ and $s = \frac{2}{m}$

If $s = 1$, we have

$$\frac{\gamma}{2+\gamma m} = m w^{1-2/m}\psi^{\gamma-1}\psi' \qquad 2+\gamma m \neq 0 \tag{4.6}$$

where by (4.4) the constant of integration is zero. Equation (4.6) integrates to

$$\psi = \left(\gamma + \frac{\gamma^2 Z^2}{1+\frac{\gamma m}{2}}\right)^{1/\gamma} \qquad 1 + \frac{\gamma m}{2} > 0. \tag{4.7}$$

If $1 + \frac{\gamma m}{2} < 0$, equation (4.7) does not hold (or the solution of (4.3)–(4.4) does not exist). If $1 + \frac{\gamma m}{2} = 0$, equation (4.5) is rewritten in the form

$$\psi + \lambda(w\psi)' = m^2(w^{2-2/m}\psi^{-2/m}\psi')'. \tag{4.8}$$

By setting $K = \int_0^w \psi \, \mathrm{d}w_1$ in (4.8), it integrates to

$$K + \lambda w K' = m^2 w^{2-2/m} K'^{-2/m} K''. \tag{4.9}$$

Equation (4.9) integrates in two cases: $\lambda = 1$, $m = 1$ and $\frac{1}{\lambda} = \frac{2}{m} - 1$, $m \neq 2$. If $\lambda = 1$ and $m = 1$, we have

$$wK = -\frac{1}{K'} + 1. \tag{4.10}$$

By eliminating K from (4.10) and (4.9), and integrating the resulting equation, we find

$$Z = \mathrm{e}^{-\frac{1}{2}\left(\frac{1-\psi}{\psi z}\right)^2} \int_0^{\frac{\psi-1}{\psi z}} \mathrm{e}^{\frac{y^2}{2}} \mathrm{d}y. \tag{4.11}$$

Similar treatment holds in the case $\lambda = -3$ and $m = 3$. If $m = 2$ in (4.9), we use the Euler transformation $w = e^t$ and find that it becomes

$$K + \lambda K^* = \frac{4}{K^*}(K^{**} - K^*) \qquad K^* = \frac{dK}{dt}. \tag{4.12}$$

This equation integrates to

$$K^* e^{-\frac{\lambda}{4}(K+4)} = -e^{-\frac{\lambda}{4}(K+4)}\left[\frac{K+4}{\lambda} + \frac{4}{\lambda^2}\right] + C_0 \tag{4.13}$$

where $C_0 = \frac{e^{-\lambda}}{\lambda^2}(\lambda + 2)$, by using (4.4). By eliminating K from (4.13) and (4.9), we obtain

$$\frac{4}{\lambda^2} + \frac{4}{\lambda} + \frac{4w\psi'}{\lambda\psi} = \frac{(\lambda+2)^2}{\lambda\psi} e^{\frac{\lambda w}{4}(\lambda\psi - 4\psi^{-1}\psi')}. \tag{4.14}$$

Note that if $\lambda = -2$, the solution of (4.14) (or (4.9)) which satisfies (4.4) does not exist.

If $s = \frac{2}{m}$ or $\frac{\lambda+1}{\gamma\lambda} = -1$, equation (4.5) integrates to

$$\frac{\gamma}{1+\gamma} w^{2/m}\psi = mw\psi^{\gamma}\psi' - (2-m)\frac{\psi^{\gamma+1}}{\gamma+1} + \frac{2-m}{\gamma+1} \tag{4.15}$$

where $\gamma + 1 \neq 0$. If $m = 2$, then (4.15) integrates to

$$\psi = \left(1 + \frac{\gamma Z^2}{2}\right)^{1/\gamma}. \tag{4.16}$$

When $m = 1$ and $\gamma = -\frac{1}{2}$, equation (4.15) integrates to

$$\psi = \frac{4}{Z^2}\left(\frac{e^z - 1}{e^z + 1}\right)^2. \tag{4.17}$$

When $\gamma = -1$, $\lambda = \frac{2}{m-2}$ and $m \neq 2$, equations (4.3)–(4.4) integrate to

$$\psi = \frac{1}{1 + \frac{Z^2}{2(M-2)}}. \tag{4.18}$$

If $\gamma = -1$, $m = 2$ and $\lambda = -\frac{1}{2}$, equations (4.3)–(4.4) integrate to

$$\psi = \frac{1}{1 + \frac{z^2}{8}}. \tag{4.19}$$

Further, we have studied partially invariant solutions of (1.6) and (1.5) but no new solutions have been found, apart from the steady-state one. The details are not, therefore, reproduced here.

Finally, we study travelling-wave solutions of (1.6) in the next section.

5. Travelling wave solutions of (1.6)

Solutions of (1.6) in the form of travelling waves propagating at a speed C are assumed in the form $N = \psi(Z)$, $Z = X \pm C\tau$. The $\pm$ signs indicate the direction of propagation of these waves.

When substituting this solution into (1.6), we find that it exists only when $m = 1$ and ψ satisfies

$$\pm C\psi' = (\psi^{\gamma}\psi') - \psi^{\beta+2}. \tag{5.1}$$

We search for solutions of (5.1) which satisfy $\psi(Z \uparrow \infty) = 0$. It can be written in the form

$$(\gamma+1)\frac{\mathrm{d}k}{\mathrm{d}h} + hk^{-\frac{(\beta+2)}{\gamma+1}} = \mp Ck^{-\frac{(\beta+1)}{\gamma+1}} \qquad \gamma+1 \neq 0 \tag{5.2}$$

where $k = \psi^{\gamma+1}$ and $h = k' \mp Ck^{-\frac{1}{\gamma+1}}$. Equation (5.2) integrates when $-\frac{(\beta+2)}{(\gamma+1)} = 1$ and $-\frac{(\beta+1)}{(\gamma+1)} = 1$. In the first case, we have

$$\psi^{-1}\mathrm{e}^{\frac{h^2}{2(\gamma+1)^2}} = \frac{\mp C}{(\gamma+1)^2}\int_h^{\infty} \mathrm{e}^{\frac{y^2}{2(\gamma+1)^2}}\,\mathrm{d}y \tag{5.3}$$

where $h = (\gamma+1)\psi^{\gamma}\psi' \mp C\psi$ and $\gamma + 1 < 0$. In the second case equation (5.1) integrates to

$$\psi = \frac{-1}{(\gamma+1)^2\alpha}(h - \frac{1}{\alpha}) - \frac{\mathrm{e}^{-\alpha h}}{\alpha^2} \tag{5.4}$$

where $\alpha = \frac{\mp C}{(\gamma+1)^2}$ and h is given above. If $\gamma + 1 < 0$, we have for travelling waves propagating in the positive direction to the x-axis the expression

$$\psi = \left[D + \frac{(\gamma+1)^2}{C^2}Z\right]^{\frac{1}{\gamma+1}} \qquad Z > \frac{-C^2D}{(\gamma+1)^2} \tag{5.5}$$

where D is a constant. We notice that further integrals of (5.2) can be found for some special values of β and $\gamma \neq -1$. Further studies of equations (3.2) and (4.3) will be carried out in future work. Invariants of these equations, quadratic and of higher orders in ψ, will be investigated in view of the results in [12, 13].

6. Conclusions

We have studied equation (1.6) using the Lie group analysis and identified similarity transformations. Among hypothesized classical similarity solutions of (1.6), we have found that if $-1 < \beta < 1$ and $-1 < \gamma < 1$, these solutions are either decreasing or increasing. Using the terminology proposed here, the values of $-1 < \beta < 1$ and $-1 < g < 1$ correspond to a plasma regime of weak nonlinearity and weak diffusivity. This situation corresponds to that found in the experiments of [1–3].

Furthermore, solutions for plasma regimes of weak (or strong) nonlinearity and weak (or strong) diffusivity have been investigated. The limiting values, $\beta = \pm 1$ and $\gamma = \pm 1$, are interesting in the sense that they separate four plasma regimes. Classical similarity solutions of (1.6) have been revealed for these values.

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